has remarked, the theorems are thus related to the lesser Fermat theorem and to Wilson's theorem which are concerned respectively with the numbers which satisfy the difference equations $u_{x+1} = au_x$, and $u_{x+1} = xu_x$.

[1] Hurwitz, *Zurich, Vierteljahrsch. Natf. Ges.*, **41,** 1896, (34–64).

[2] Cotes, *Phil. Trans., London,* **29,** 1714, (5). (Reference in *Encyc. Sci. Math.*, Paris, Tome I, Vol. I, Fasc. 2, p. 169 note.)

[3] Euler, *Com. Acad. Petropolitanae,* **9,** 1737, (98–137).

[4] Tietze, *Math. Ann., Leipzig,* **70,** 1911, (236–265).

ON CLOSED CURVES DESCRIBED BY A SPHERICAL PENDULUM

By Arnold Emch

Department of Mathematics University of Illinois

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1. The geometrical aspect of the problem of closed curves in a spherical pendulum-motion has apparently never been fully discussed and it is the object of this note to present the results of an investigation of some of the geometric properties of these curves.

The differential equations of the motion are

$$\frac{d^2x}{dt^2} = -S\frac{x}{l},\quad \frac{d^2y}{dt^2} = -S\frac{y}{l},\quad \frac{d^2z}{dt^2} = -S\frac{z}{l} - g, \tag{1}$$

in which l is the length of the pendulum, S the variable reaction directed towards the origin (point of suspension). The velocity v of the pendulum-bob is, as is well known, given by $v^2 = h - 2gz$, in which h is a constant depending upon the initial conditions. For z as a function of t, we may obtain without difficulty

$$z = \frac{h}{6g} + \frac{2l^2}{g} p\ (t + w_2). \tag{2}$$

In this Weierstrassian p-function the one half-period w_1 is real; the other w_2 is pure imaginary. Putting $u = t + w_2$, let a and b be the arguments for which z assumes the values $-\ l$ and $+\ l$, respectively, and t_1 and t_2 the corresponding complex values of t. It is found that $a = i\ \alpha$, where α is real and positive, and $b = w_1 + i\beta$, with β equal to a fraction of $|\ w_2\ |$. The constant of integration may be determined such that $x = r_0$, $y = 0$, when $t = 0$. Under these conditions $\theta = \tan^{-1}(y/x)$ is defined by

$$e^{2i\theta} = \frac{\sigma(t+t_1)\,\sigma(t+t_2)}{\sigma(t-t_1)\,\sigma(t-t_2)} \cdot e^{-2t[\zeta_2(t_1)+\zeta_2(t_2)]} \tag{3}$$

From this is found

$$x = \phi(t) = \frac{r_0}{2} \frac{\sigma(t+t_1)\,\sigma(t+t_2)\,e^{-t[\zeta_2(t_1)+\zeta_2(t_2)]} + \sigma(t-t_1)\,\sigma(t-t_2)\,e^{t[\zeta_2(t_1)+\zeta_2(t_2)]}}{\sigma(t_1)\,\sigma(t_2)\,\sigma_2^2(t)}, \tag{4}$$

$$y = x(t) = -\frac{r_2 i}{2} \cdot \frac{\sigma(t+t_1)\,\sigma(t+t_2)\,e^{-t[\zeta_2(t_1)+\zeta_2(t_2)]} - \sigma(t-t_1)\,\sigma(t-t_2)\,e^{t[\zeta_2(t_1)+\zeta_2(t_2)]}}{\sigma(t_1)\,\sigma(t_2)\,\sigma_2^2(t)}. \tag{5}$$

These solutions may also be obtained from the differential equations.

$$\frac{d^2x}{dt^2} = \left[6\,p(t) - \frac{h}{2l^2} \right] x, \tag{6}$$

and a similar equation for y. (4) and (5) are uniform analytic functions of t, with ∞ as the only essential irregularity, and they assume real values for real values of t.

Increasing t by $2\,m_1\,w_1$, it is found that the motion will be periodic, when

$$2\,m_1\{\eta_1(a+b) - w_1[\xi(a) + \xi(b)]\} = 2ki\pi, \tag{7}$$

k being a positive integer. When t increases by $2\,w_1$, θ will increase by the amount

$$\Phi = -2i\{\eta_1(a+b) - w_1[\xi(a) + \xi(b)]\}, \tag{8}$$

so that in case of a periodic motion, from (7) and (8),

$$\Phi = 2k\pi/m_1. \tag{9}$$

The path described by the pendulum is now a closed curve intersecting every level between the lowest and highest position in 2 m_1 points and winding k times around the z-axis, before it closes.

Moreover a function-theoretic investigation of the periodic function

$$F(t) = \alpha.\phi(t) + \beta.\psi(t) + \gamma, \tag{10}$$

whose zeros determine the intersections of the straight line $\alpha x + \beta y + \gamma = 0$ with the horizontal projection of the curve (x, y), shows that *the curve is algebraic and passes through the circular (isotropic) points at infinity (i.e., its horizontal projection)*. From (3) follows that *the curve has an m_1-fold axial symmetry.*

2. In Greenhill's case[1] the pendulum-bob reaches (but does not go above) the horizontal plane of suspension with a non-vanishing horizontal velocity, and the parametric expressions for the horizontal projection may be written in the form:[2]

$$x = r_0[\cos (A + \pi) v. \operatorname{cn} (2 Kv) - \mu \sin (A + \pi) v. \operatorname{sn} (2 Kv) \operatorname{dn} (2 Kv)], \quad (11)$$

$$y = r_0[\sin (A + \pi) v. \operatorname{cn} (2 Kv) + \mu \cos (A + \pi) v. \operatorname{sn} (2 Kv) \operatorname{dn} (2 Kv)]. \quad (12)$$

If $x' = r_0 \operatorname{cn} (2 Kv)$, $y' = \mu r_0 \operatorname{sn} (2 Kv) \operatorname{dn} (2 Kv)$, $z' = a \operatorname{cn}^2 (2 Kv)$, (13)

it is found that *the locus of the point $P'(x', y', z')$ is a quartic C_4 in space, and that P' is obtained by rotating the point P (x, y, z) of the pendulum curve C about the z-axis through an angle $-(A + \pi) v$, i.e., through an angle negatively proportional to the time associated with P in the motion.*

Imposing upon (11) and (12) the same condition of periodicity, as in the general case, *the resulting curve in the xy-plane becomes a rational algebraic curve of order $2 k$. The angle corresponding to the period $2 w_1$ is, as before, $\Phi = 2k\pi/m_1$; k and m_1 being relatively prime. For a given odd m_1 there are $(m_1 - 1)/2$ curves of this type, for an even m_1, there are $(m_1 - 2)/2$ such curves. In both cases there are $m_1(k - 1)$ real double points.*

When m_1 is odd there is just one curve among the set whose double-points are all real. Its degree is $2 k = m_1 + 1$.

When m_1 is even, there is no such curve.

In case of an odd m_1 the polar equation of the curve has the form

$$\begin{aligned} &\rho^{2k} + a_1\rho^{2k-2} + a_2\rho^{2k-4} + \dots + a_j\rho^2 + a + \\ &(b_1\rho^{2\lambda_1} + b_2\rho^{2\lambda_2} + \dots + b_j\rho^2 + b)\ \rho^{m_1} \cos m_1\theta = 0, \end{aligned} \quad (14)$$

in which $a \pm 0$, $\lambda_1 > \lambda_2 > \lambda_3 > \dots > 1$, and $2 \lambda_1 + m_1 \leqq 2 k - 1$. In cartesian coordinates (14) may be written in the form

$$\begin{aligned} &(x^2 + y^2)^k + a_1 (x^2 + y^2)^{k-1} + a_2 (x^2 + y^2)^{k-2} + \dots + a_j (x^2 + y_2) + a + \\ &\quad \left\{b_1 (x^2 + y^2)^{\lambda_1} + b_2 (x^2 + y^2)^{\lambda_2} + \dots + b_j (x^2 + y^2) + b\right\}. \\ &\left\{x^{m_1} - \binom{m_1}{2} x^{m_1-2} y^2 + \binom{m_1}{4} x^{m-4} y^4 + \dots + (-1)^{\frac{m_1-1}{2}} \cdot xy^{m_1-1}\right\} = 0. \end{aligned} \quad (15)$$

Transforming this curve by

$$x = \pm \sqrt{x'' - y''^2 - 2y'' - 1} / y'', \; y = (y'' + 1) / y'',$$

or using isotropic coordinates, transformations which do not change the character of the isotropic points, and "placing the curve on the analytic triangle," the result is obtained that the isotropic points absorb together

$$(k - 1)(2 k - m_1 - 1) \quad (16)$$

double points, which when added to the $m_1 (k - 1)$ real double-points, gives $(k - 1) (2 k - 1)$, i.e., the maximum number of double points which a curve of order $2 k$ may have. Thus we have verified directly from the equation, that the curve is rational, as proved before. When the curve has all double points real, then its degree is $m_1 + 1 = 2 k$, so that from (16) the number of

imaginary double points is clearly zero, which is another verification of previous results. (16) is also true when m_1 is even. Polar and cartesian equations for m_1 even may be established in a similar manner as in (14) and (15).

3. Examples.—When $m_1 = 3$, $2k = 4$, the polar and rational parametric equations of the curve[3] are

$$\rho^4 - R\rho^3 \cos 3\theta - \frac{27}{4} R^2\rho^2 + \frac{27}{4} R^4 = 0$$

$$x = \frac{t^4 - 12t^2 + 3}{4(1+t^2)^2},\ y = \frac{2t(3-5t^2)}{4(1+t^2)^2};\ (R = \tfrac{1}{4})$$

When $m_1 = 4$, $2k = 6$, cartesian and polar equations of the curve are

$$3(x^2 + y^2)^3 - 24(x^4 + y^4) - 32\ x^2y^2 + 39(x^2 + y^2) - 18 = 0,$$
$$3\rho^6 - (2\cos 4\theta + 22)\rho^4 + 39\rho^2 - 18 = 0.$$

As m and k increase, the construction of the equations with numerical coefficients becomes increasingly difficult.[4]

[1] Greenhill, *Les Fonctions Elliptiques et leurs Applications*, chap. III.

[2] Tannery et Molk, *Eléments de la Théorie des Fonctions Elliptiques*, **4**, pp. 176–192; Appell: *Traité de Mécanique Rationnelle*, **1**, p. 494.

[3] This curve is known and was investigated by G. de Longchamps: *J. Math. Elémentaires*, **4**, 1885, 269–277; also by H. Brocard: *J. Math. Spéciales*, (Ser. 3), **5**, 1891, 56–64.

[4] A curve of this type may be symbolically denoted by $C_{2k}^{m_1}$. Tabulations of all curves, with their real and imaginary double points as far as C_{22}^{12} have been made, also actual graphs of C_4^3, C_4^6, C_6^5. C_8^5.

THE TAXONOMIC POSITION OF THE GENUS ACTINOMYCES

By Charles Drechsler

Cryptogamic Laboratories, Harvard University

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To the genus Actinomyces are usually assigned a variety of peculiar and very minute filamentous organisms, widely distributed in nature, concerning the taxonomic relationship of which divergent views have been held. Most of the earlier medical writers, whose attention was centered on forms associated with human and animal diseases, placed the genus with the pleomorphic bacteria. Others recognized conidia in the clavate elements produced by one of these pathogenic forms, *Act. bovis*, within the animal body, and referred this parasite to the Fungi. These elements were later shown to represent degenerative structures; but as subsequent investigations on a number of species, including various saprophytic forms, as well as the common potato scab